



UNIVERSITY OF
ILLINOIS LIBRARY
AT URBANA-CHAMPAIGN
BOOKSTACKS

Digitized by the Internet Archive
in 2011 with funding from
University of Illinois Urbana-Champaign

<http://www.archive.org/details/deterministicthe52ande>

Faculty Working Papers

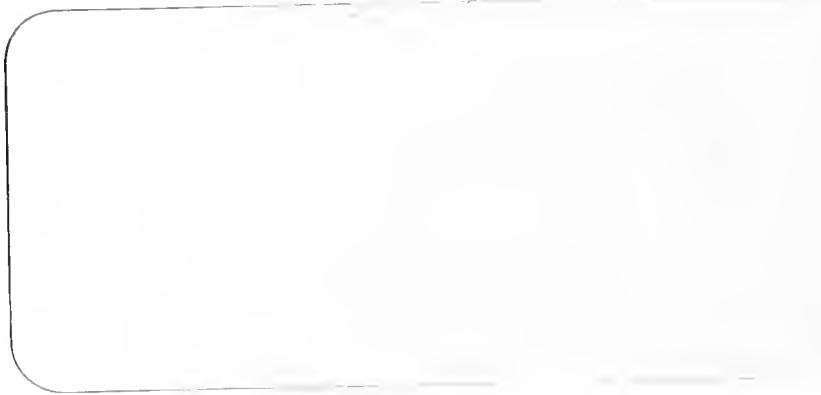
A Deterministic Theory of Individual Saving
and Portfolio Composition

Robert E. Anderson

University of Illinois

#52

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign



FACULTY WORKING PAPERS

College of Commerce and Business Administration

April 27, 1972

A Deterministic Theory of Individual Saving
and Portfolio Composition

Robert E. Anderson

University of Illinois

#52

Department of Mathematics, University of Cambridge, Cambridge, UK

(Received 10 January 2012)

NON-DISTRIBUTABLE

NON-DISTRIBUTABLE

FIELDS

Introduction

The goal of this research is to develop a model that will show how an individual chooses his lifetime patterns of consumption and saving and how he simultaneously chooses the composition of his portfolio of assets at each point during his lifetime. The individual is assumed to know with certainty his future pattern of income, the rates of interest that he can earn on the various assets in his portfolio, and the rates of interest that he must pay if he goes into debt. The theory of portfolio composition presented here is not based on the usual trade-off between the expected yield and the riskiness of an asset. Instead it is based on the trade-off between the yield or the rate of interest that can be earned on each asset and the transactions costs involved in buying and selling that asset.

An asset with a low rate of interest and low transactions costs (for example, savings accounts) will only be held if the surplus is to last for a relatively short period of time. The individual is compensated for the low interest rate by the small costs involved in buying and selling that asset. If the surplus is to last for a relatively long period of time, an individual is better off holding the surplus in the form of an asset with a higher interest rate and undoubtedly higher transactions costs (for example, corporate bonds, real estate, etc.) Similarly at those points in an individual's lifetime when he is a net debtor, he will choose the type of loan to finance the deficit by comparing the rate of interest that he must pay against the cost of taking out and

paying back the loan. If his indebtedness is to last for a relatively short period of time, he is better off taking out a loan which has low transactions costs even though he will probably have to pay a higher rate of interest. For example, these loans might be in the form of charge accounts or credit cards. If the deficit is to last for a long period of time, he would be better off arranging for a loan with lower interest charges and undoubtedly higher transactions costs (for example, mortgages or other secured loans).

What this model will show is that if the individual is in a surplus position then he will probably be holding a number of different assets each with a different interest rate and transactions cost. In other words, at each point in time the individual's portfolio will usually contain a variety of different assets. Similarly, if the individual is in a net deficit position then he will have financed this deficit by taking out a variety of different types of loans each with a different interest rate and transactions costs. At each point during his lifetime, the individual will either be in a surplus position or a deficit position depending on whether his cumulative expenditures for consumption is greater than or less than his cumulative income up to that point. For simplicity, it is assumed that the individual does not borrow in order to buy an asset.

The Individual's Optimization Problem

An individual is assumed to maximize the discounted sum of instantaneous utility over his lifetime, i.e.,

$$(1) \quad \max \int_0^T e^{-\theta t} u(c) dt$$

• (cont'd.)

For more information about the study, please contact Dr. Michael J. Koenig at (314) 747-2000 or via email at koenig@dfci.harvard.edu.

—
—
—

1. *U.S. Fish and Wildlife Service*, 1871-1940, *U.S. Fish Commission*, 1871-1890.

J *I* *R* *S*

Report of the Bureau of Fisheries

As a result, the model is able to learn the underlying structure of the data and make predictions for new data points based on the learned patterns.

1. *Proteins* 2. *Carbohydrates* 3. *Lipids* 4. *Nucleic acids*

11. *Chlorophytum comosum* (L.) Willd.

As a result, the number of people who have been infected with the virus has increased rapidly.

¹⁰ See also the discussion of the relationship between the concept of ‘cultural capital’ and the concept of ‘cultural value’ in the section on ‘Cultural Capital’ above.

¹ See also the discussion of the relationship between the two concepts in the section on "The Concept of Social Capital."

For more information about the study, please contact Dr. Michael J. Hwang at (319) 356-4000 or email at mhwang@uiowa.edu.

and the author's name is omitted from the title page.

$$(\beta) \quad g^{(k)} = \frac{1}{\beta} \chi_{\{g_k > 0\}}$$

where θ is his constant discount rate, $c(t)$ is his lifetime pattern of consumption, and $u(c)$ is a differentiable and concave utility function. The price level is constant and equal to one. His lifetime pattern of endowment income is denoted by $x(t)$, $0 \leq t \leq T$.

For simplicity, assume that there are only two types of assets or bonds that he may hold, denoted by $B_1(t)$ and $B_2(t)$. Each type of bond earns a rate of interest, denoted by r_1 and r_2 respectively; and these rates are constant over time. Similarly the individual may take out two different types of loans where the total indebtedness of each type is denoted by $D_1(t)$ and $D_2(t)$. On each type of debt, he must pay a rate of interest, denoted by i_1 and i_2 respectively, which are also constant over time. The stocks of bonds and the stocks of debt must be non-negative.

At any point in time, an individual's total receipts from all sources must equal his total expenditures. An individual's total receipts is the sum of his endowment income, interest income earned on positive stocks of bonds, receipts from the sale of bonds of either type, and the receipts from taking out additional loans of either type. His total expenditures is the sum of consumption expenditures, interest payments on his debts, expenditures for the purpose of adding to his stocks of bonds of either type, and expenditures for repaying loans of either type. The receipts from selling bonds are denoted by S_1 and S_2 , and the receipts from taking out additional loans are denoted by L_1 and L_2 . Expenditures for increasing the stocks of bonds are denoted by A_1 and A_2 , and expenditures for the repayment of loans are denoted by R_1 and R_2 . His budget constraint at each point in time is thus

$$(2) \quad x + r_1 B_1 + r_2 B_2 + S_1 + S_2 + L_1 + L_2$$

$$- c - i_1 D_1 - i_2 D_2 - A_1 - A_2 - R_1 - R_2 = 0$$



Fig. 1. Water density distribution function $\rho(r)$ versus distance r : (—) water-water ($\rho_{ww}(r)$) and (—) water-polymer ($\rho_{wp}(r)$).

the water density distribution function $\rho(r)$ versus distance r is shown in Fig. 1. The water density distribution function $\rho(r)$ is defined as the probability density of finding a water molecule at a distance r from the center of a polymer molecule. The water density distribution function $\rho(r)$ is calculated by the equation

$$\rho(r) = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^{N_i} \delta(r - r_{ij}) \quad (1)$$

where N is the total number of water molecules, N_i is the number of water molecules in the i th sphere, r_{ij} is the distance between the i th water molecule and the center of the j th polymer molecule, and δ is the Dirac delta function.

The water density distribution function $\rho(r)$ is plotted in Fig. 1. The water density distribution function $\rho(r)$ is calculated for the water-water correlation function $\rho_{ww}(r)$ and the water-polymer correlation function $\rho_{wp}(r)$. The water density distribution function $\rho(r)$ is calculated for the water-water correlation function $\rho_{ww}(r)$ and the water-polymer correlation function $\rho_{wp}(r)$.

The water density distribution function $\rho(r)$ is plotted in Fig. 1. The water density distribution function $\rho(r)$ is calculated for the water-water correlation function $\rho_{ww}(r)$ and the water-polymer correlation function $\rho_{wp}(r)$.

The water density distribution function $\rho(r)$ is plotted in Fig. 1. The water density distribution function $\rho(r)$ is calculated for the water-water correlation function $\rho_{ww}(r)$ and the water-polymer correlation function $\rho_{wp}(r)$.

The water density distribution function $\rho(r)$ is plotted in Fig. 1. The water density distribution function $\rho(r)$ is calculated for the water-water correlation function $\rho_{ww}(r)$ and the water-polymer correlation function $\rho_{wp}(r)$.

On any changes in his stocks of bonds or debt, the individual must pay transactions costs proportional to the size of the change. The cost of changing the size of the stock of each type of bond by one dollar is given by v_1 and v_2 respectively. The cost of changing the size of the stock of each type of debt by one dollar is given by w_1 and w_2 respectively. Thus, the net change in the stock of each type of bond and debt is given by

$$(3) \quad B_1 = (1 - v_1)A_1 - (1 + v_1)S_1$$

$$(4) \quad B_2 = (1 - v_2)A_2 - (1 + v_2)S_2$$

$$(5) \quad D_1 = (1 + w_1)L_1 - (1 - w_1)R_1$$

$$(6) \quad D_2 = (1 + w_2)L_2 - (1 - w_2)R_2$$

where

$$(7) \quad B_1, B_2, D_1, D_2, A_1, S_1, A_2, S_2, L_1, R_1, L_2, R_2, \geq 0$$

The individual is assumed to start his lifetime with initial stocks of bonds and initial stocks of debt which may or may not be positive, i.e.

$$(8) \quad B_1(0) = B_1^0, \quad B_2(0) = B_2^0$$

$$(9) \quad D_1(0) = D_1^0, \quad D_2(0) = D_2^0.$$

At the end of his lifetime, the individual is assumed to have no desire to leave an inheritance. It is also assumed that his creditors will not allow him to leave any outstanding debts. Therefore,

$$(10) \quad B_1(T) = B_2(T) = D_1(T) = D_2(T) = 0$$

There are two features of this optimization problem that make it different from conventional problems in optimal control. The first is the inequality constraints on the state variables and the control variables given by condition (7). The second and much more important feature is

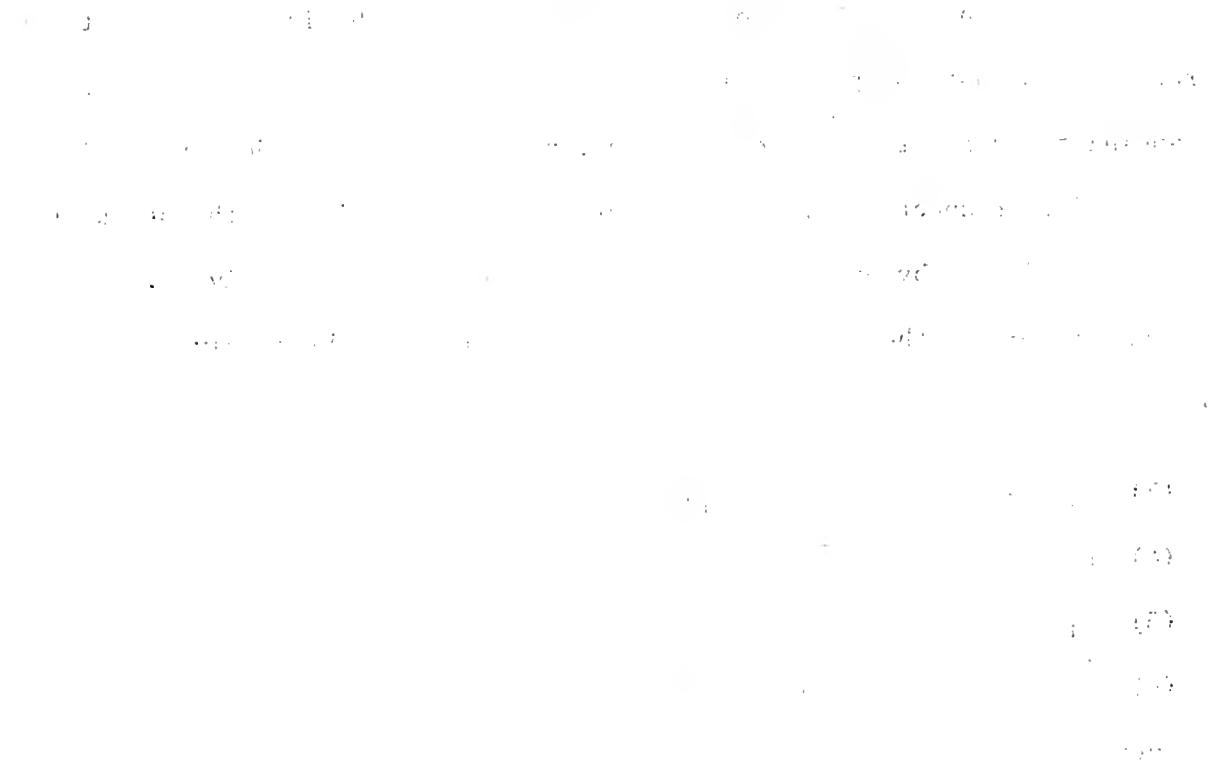


Fig. 1. Evolution of the parameter α_1 over time t for different initial values of α_1 .

3.1. Initial values of α_1 and the approach to steady state

We first investigate the effect of the initial value of α_1 on the steady-state value.

Figure 1 shows the evolution of α_1 over time t for different initial values of α_1 .

$$\begin{aligned} \alpha_1(0) &= 0.01 & \alpha_1(0) &= 0.05 & \alpha_1(0) &= 0.1 \\ \alpha_1(t) &= 0.01 & \alpha_1(t) &= 0.05 & \alpha_1(t) &= 0.1 \end{aligned}$$

It is observed that the steady-state value of α_1 is independent of the initial value of α_1 .

3.2. Effect of the initial values of α_2 and α_3 on the steady-state value

Figure 2 shows the evolution of α_1 over time t for different initial values of α_2 and α_3 .

$$\begin{aligned} \alpha_1(0) &= 0.01 & \alpha_1(0) &= 0.05 & \alpha_1(0) &= 0.1 \\ \alpha_2(0) &= 0.01 & \alpha_2(0) &= 0.05 & \alpha_2(0) &= 0.1 \\ \alpha_3(0) &= 0.01 & \alpha_3(0) &= 0.05 & \alpha_3(0) &= 0.1 \end{aligned}$$

It is observed that the steady-state value of α_1 is independent of the initial values of α_2 and α_3 .

Figure 3 shows the evolution of α_1 over time t for different initial values of α_2 and α_3 . It is observed that the steady-state value of α_1 is independent of the initial values of α_2 and α_3 .

that the time path of the state variables may be discontinuous. In other words, it is possible in this model that the optimal stocks of bonds or debt may make instantaneous jumps in size.

For example, when might it be optimal for the individual to suddenly sell a part of his stock of one type of bond and use the proceeds to buy another type of bond or to pay off a debt? It is not difficult to convince oneself that this could never be optimal except at the initial time point, $t = 0$. If such jumps were to occur later in his lifetime, the individual must have made a mistake sometime in the past or his expectations of future income or interest rates have changed. If it is now optimal for an individual to suddenly sell one type of bond in order to buy another type then the question is why was this not done earlier when the surplus was accumulated in order to save transactions costs. However, the initial stocks of bonds and debt with which the individual begins his lifetime are assumed to be given to him exogenously, and it may be optimal to make an initial rearrangement of the composition of his portfolio of assets and debts.

To allow for this initial rearrangement of the individual's portfolio, let the initial changes in the stocks of bonds or debt be denoted by \hat{A}_1 , \hat{S}_1 , \hat{A}_2 , \hat{S}_2 , \hat{L}_1 , \hat{R}_1 , \hat{L}_2 , and \hat{R}_2 . The new stocks after the initial rearrangement are denoted by $B_1(0^+)$, $B_2(0^+)$, $D_1(0^+)$, and $D_2(0^+)$. The new values of the stocks are related to the old by

$$(11) \quad B_1(0^+) = B_1(0) + (1 - v_1)\hat{A}_1 - (1 + v_1)\hat{S}_1$$

$$(12) \quad B_2(0^+) = B_2(0) + (1 - v_2)\hat{A}_2 - (1 + v_2)\hat{S}_2$$

$$(13) D_1(0^+) = D_1(0) + (1 + v_1) \hat{L}_1 - (1 - w_1) \hat{R}_1$$

$$(14) D_2(0^+) = D_2(0) + (1 + v_2) \hat{L}_2 - (1 - w_2) \hat{R}_2$$

where as always transactions costs must be paid on any changes in a stock.

Any change in the value of one stock must be matched by an appropriate change in another stock, i.e.

$$(15) \hat{A}_1 - \hat{S}_1 + \hat{A}_2 - \hat{S}_2 - \hat{L}_1 + \hat{R}_1 - \hat{L}_2 + \hat{R}_2 = 0$$

where

$$(16) \hat{A}_1, \hat{S}_1, \hat{A}_2, \hat{S}_2, \hat{L}_1, \hat{R}_1, \hat{L}_2, \text{ and } \hat{R}_2 \geq 0$$

In conclusion, the individual's goal is to find the pattern of consumption, the patterns of bond holdings, and the patterns of indebtedness over his lifetime that maximize (1) subject to the constraints (2) through (16). Because of the non-negativity constraints on both the state variables and the control variables and the possibility of discontinuities in the state variables, it does not seem possible to use the conventional theory of optimal control to derive necessary conditions for an optimum for this problem. However, it is possible to state sufficient conditions for an optimum.

Sufficient Conditions for an Optimum

It is possible to give conditions that if satisfied will guarantee that the optimal solution to the above problem has been found. These conditions involve five shadow prices, $\eta(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$. The first shadow price, η , is equal to the discounted value of the marginal utility of consumption. The variables λ_1 and λ_2 are the implicit prices that the individual places on the two stocks of bonds. The variables μ_1 and μ_2 are the implicit prices that the individual places on the two stocks of debt.

$$\frac{1}{\sqrt{2}}(1, -1, 0, 0) = \frac{1}{\sqrt{2}}(1, 0, 0, 0) + \frac{1}{\sqrt{2}}(0, 0, 1, 0)$$

- 2 -

111

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial x_i} \right) + \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j} \right) \delta_{ij}$$

For more information, contact the Office of the Vice President for Research at (319) 335-1231 or visit www.vpr.iastate.edu.

For more information, visit www.ams.org.

* (1972), 1.

The statement of these conditions is as follows: If there exists functions $c(t)$, $A_1(t)$, $S_1(t)$, $A_2(t)$, $S_2(t)$, $L_1(t)$, $R_1(t)$, $L_2(t)$, and $R_2(t)$ continuous for $0 \leq t \leq T$, functions $B_1(t)$, $B_2(t)$, $D_1(t)$ and $D_2(t)$ continuous for $0 < t \leq T$, and values for \hat{A}_1 , \hat{S}_1 , \hat{A}_2 , \hat{S}_2 , \hat{L}_1 , \hat{R}_1 , \hat{L}_2 , and \hat{R}_2 all satisfying relations (2) through (16) and functions $\eta(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ continuous for $0 \leq t \leq T$ where \dot{B}_1 , \dot{B}_2 , \dot{D}_1 , \dot{D}_2 , $\dot{\eta}$, $\dot{\lambda}_1$, $\dot{\lambda}_2$, $\dot{\mu}_1$, and $\dot{\mu}_2$ are integrable such that at each point in time

$$(17) \quad \eta = e^{-\theta t} u'(c)$$

$$(18) \quad \underline{\int} \dot{\lambda}_j + r_j \eta \overline{\int} B_j = 0 \quad j = 1, 2$$

$$(19) \quad \dot{\lambda}_j + r_j \eta \leq 0 \quad j = 1, 2$$

$$(20) \quad \underline{\int} (1 - v_j) \lambda_j - \eta \overline{\int} A_j = 0 \quad j = 1, 2$$

$$(21) \quad (1 - v_j) \lambda_j - \eta \leq 0 \quad j = 1, 2$$

$$(22) \quad \underline{\int} \eta - (1 + v_j) \lambda_j \overline{\int} S_j = 0 \quad j = 1, 2$$

$$(23) \quad \eta - (1 + v_j) \lambda_j \leq 0 \quad j = 1, 2$$

$$(24) \quad \underline{\int} \dot{\mu}_j - i_j \eta \overline{\int} D_j = 0 \quad j = 1, 2$$

$$(25) \quad \dot{\mu}_j - i_j \eta \leq 0 \quad j = 1, 2$$

$$(26) \quad \underline{\int} (1 + w_j) \mu_j + \eta \overline{\int} L_j = 0 \quad j = 1, 2$$

$$(27) \quad (1 + w_j) \mu_j + \eta \leq 0 \quad j = 1, 2$$

$$(28) \quad \underline{\int} \eta - (1 - w_j) \mu_j \overline{\int} R_j = 0 \quad j = 1, 2$$

$$(29) \quad \eta - (1 - w_j) \mu_j \leq 0 \quad j = 1, 2$$

and at time $t = 0$

$$(30) \quad \underline{\int} (1 - v_j) \lambda_j - \eta \overline{\int} \hat{A}_j = 0 \quad j = 1, 2$$

$$(31) \quad (1 - v_j) \lambda_j - \eta \leq 0 \quad j = 1, 2$$

$$(32) \quad \underline{\int} \eta - (1 + v_j) \lambda_j \overline{\int} \hat{S}_j = 0 \quad j = 1, 2$$

$$(33) \quad \eta - (1 + v_j) \lambda_j \leq 0 \quad j = 1, 2$$

卷之三

1. *Leucosia* *leucostoma* *lutea*

$$\{Y_i\}_{i=1}^n \sim \text{Uniform}(\{1, 2, \dots, k\})$$

© 2019 by the author. Licensee MDPI, Basel, Switzerland.

As a result, the number of patients with a history of stroke or TIA was significantly higher in the group with a history of hypertension than in the group without a history of hypertension.

$$(34) \quad \underline{L}(1 + v_j)u_j + \eta \overline{J} \hat{L}_j = 0 \quad j = 1, 2$$

$$(35) \quad (1 + w_j)u_j + \eta \leq 0 \quad j = 1, 2$$

$$(36) \quad \underline{L} - (1 - v_j)u_j \overline{J} \hat{R}_j = 0 \quad j = 1, 2$$

$$(37) \quad -\eta - (1 - w_j)u_j \leq 0 \quad j = 1, 2$$

then $c(t)$, $B_1(t)$, $B_2(t)$, $D_1(t)$, $D_2(t)$, $A_1(t)$, $S_1(t)$, $A_2(t)$, $S_2(t)$, $L_1(t)$, $R_1(t)$, $L_2(t)$, $R_2(t)$, \hat{A}_1 , \hat{S}_1 , \hat{A}_2 , \hat{S}_2 , \hat{L}_1 , \hat{R}_1 , \hat{L}_2 , and \hat{R}_2 will maximize the functional (1) subject to the conditions (2) through (16).

Proof

If we denote by an asterisk those functions and variables which satisfy the previously stated conditions then for any other feasible set of functions and variables it must be shown that

$$(38) \quad \int_0^T e^{-\theta t} u(c^*) dt \geq \int_0^T e^{-\theta t} u(c) dt$$

or that

$$(39) \quad \int_0^T e^{-\theta t} \underline{L} u(c^*) - u(c) \overline{J} dt \geq 0$$

The following string of equalities and inequalities will prove this result. The explanation of each step is given in brackets.

$$(40) \quad \int_0^T e^{-\theta t} \underline{L} u(c^*) - u(c) \overline{J} dt$$

\underline{L} since $u(c)$ is assumed to be concave \overline{J}

$$(41) \quad \geq \int_0^T e^{-\theta t} u'(c^*)(c^* - c) dt$$

\underline{L} from equation (17) \overline{J}

$$(42) \quad = \int_0^T \eta(c^* - c) dt$$

\underline{L} from equation (2) \overline{J}

13. *Leucosia* *leucostoma* (Fabricius) *Leucosia leucostoma* (Fabricius)

（三）在本办法施行前，已经完成的项目，不再进行审计。

10. V. C. FORBES, p. 17

Journal of Health Politics, Policy and Law, Vol. 35, No. 3, June 2010
DOI 10.1215/03616878-35-3 © 2010 by The University of Chicago

$$(\pm \sin \theta) M_2 = 0 \quad \text{and} \quad (\pm \cos \theta)$$

- 1 -

¹ See also the discussion of the relationship between the two concepts in the section on "The Concept of Social Capital."

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n} \right)^k = 0$$

$$(\{1\} \cup \dots \cup \{n\}) \otimes \mathcal{O}(G) \otimes \mathbb{F}_q[[\lambda]]$$

$$(43) = \int_0^T \eta \bar{S}_1^* + S_2^* + L_1^* + L_2^* + r_1 B_1^* + r_2 B_2^* - i_1 D_1^* - i_2 D_2^*$$

$$- A_1^* - A_2^* - R_1^* - R_2^* - S_1 - S_2 - L_1 - L_2 - r_1 B_1$$

$$- r_2 B_2 + i_1 D_1 + i_2 D_2 + A_1 + A_2 + R_1 + R_2 \bar{J} dt$$

/from conditions (20) through (29)

$$(44) \geq \int_0^T \bar{(1+v_1)\lambda_1 S_1^*} + (1+v_2)\lambda_2 S_2^* - (1+w_1)\mu_1 L_1^* - (1+w_2)\mu_2 L_2^* + r_1 \eta B_1^*$$

$$+ r_2 \eta B_2^* - i_1 \eta D_1^* - i_2 \eta D_2^* - (1-v_1)\lambda_1 A_1^* - (1-v_2)\lambda_2 A_2^*$$

$$+ (1-w_1)\mu_1 R_1^* + (1-w_2)\mu_2 R_2^* - (1+v_1)\lambda_1 S_1 - (1+v_2)\lambda_2 S_2$$

$$+ (1+w_1)\mu_1 L_1 + (1+w_2)\mu_2 L_2 - r_1 \eta B_1 - r_2 \eta B_2 - r_2 \eta B_2 + i_1 \eta D_1$$

$$+ i_2 \eta D_2 + (1-v_1)\lambda_1 A_1 + (1-v_2)\lambda_2 A_2 - (1-w_1)\mu_1 R_1$$

$$- (1-w_2)\mu_2 R_2 \bar{J} dt$$

/from conditions (3) through (6)

$$(45) = \int_0^T \bar{-\lambda_1 B_1^*} + \dot{\lambda_1} B_1 - \lambda_2 B_2^* + \dot{\lambda_2} B_2 + r_1 \eta B_1^* + r_2 \eta B_2^* - r_1 \eta B_1$$

$$- r_2 \eta B_2 - \mu_1 D_1^* + \dot{\mu_1} D_1 - \mu_2 D_2^* + \dot{\mu_2} D_2 - i_1 \eta D_1^* - i_2 \eta D_2^*$$

$$+ i_1 \eta D_1 + i_2 \eta D_2 \bar{J} dt$$

/integrating by parts

$$(46) = -\lambda_1 B_1^* \Big|_0^T + \int_0^T (\dot{\lambda_1} + r_1 \eta) B_1^* dt + \lambda_1 B_1 \Big|_0^T + \int_0^T (-\dot{\lambda_1} - r_1 \eta) B_1 dt$$

1

For the first time, the results of the 1990 census are available in a single, easy-to-use CD-ROM.

($\lambda = \mu$) \text{ and } (\lambda = \mu) \text{ are called two mostly}

¹⁰ A. S. K. Chiu, "The Impact of the Internet on the Chinese Society," *Journal of Computer Information Systems*, Vol. 45, No. 1, Jan/Feb 2005, pp. 1-10.

$$\left(\frac{1}{2} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \partial_{\mu} \partial_{\nu}\right) \left(\frac{1}{2} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \partial_{\mu} \partial_{\nu}\right) = \frac{1}{2} \left((\partial^{\mu} + D)^2 - (\partial^{\mu} - D)^2\right)$$

1. *Chlorophytum comosum* (L.) Willd.

$$(\partial_t^{\alpha} \partial_x^{\beta} u)(t,x) = \frac{(-1)^{\alpha}}{\Gamma(\alpha+1)} \int_0^t \partial_x^{\beta} u(t-s,x) ds + \frac{(-1)^{\alpha}}{\Gamma(\alpha+1)} \int_0^t \partial_x^{\beta} \partial_t^{\alpha} u(s,x) ds.$$

$$\begin{aligned}
 & -\lambda_2 B_2^* |_0^T + \int_0^T (\dot{\lambda}_2 + r_2 n) B_2^* dt + \lambda_2 B_2 |_0^T + \int_0^T (-\dot{\lambda}_2 - r_2 n) B_2 dt \\
 & -\mu_1 D_1^* |_0^T + \int_0^T (\dot{\mu}_1 - i_1 n) D_1^* dt + \mu_1 D_1 |_0^T + \int_0^T (-\dot{\mu}_1 + i_1 n) D_1 dt \\
 & -\mu_2 D_2^* |_0^T + \int_0^T (\dot{\mu}_2 - i_2 n) D_2^* dt + \mu_2 D_2 |_0^T + \int_0^T (-\dot{\mu}_2 + i_2 n) D_2 dt
 \end{aligned}$$

/from conditions (18), (19), (24), (25) and (10)

$$\begin{aligned}
 (47) \geq & + \lambda_1(0) B_1^*(0^+) - \lambda_1(0) B_1(0^+) + \lambda_2(0) B_2^*(0^+) - \lambda_2(0) B_2(0^+) \\
 & + \mu_1(0) D_1^*(0^+) - \mu_1(0) D_1(0^+) + \mu_2(0) D_2^*(0^+) - \mu_2(0) D_2(0^+)
 \end{aligned}$$

/from conditions (8), (9), (11), (12), (13), and (14)/

$$\begin{aligned}
 (48) = & (1 - v_1) \hat{\lambda}_1(0) \hat{A}_1^* - (1 + v_1) \hat{\lambda}_1(0) \hat{S}_1^* - (1 - v_1) \lambda_1(0) \hat{A}_1 + (1 + v_1) \lambda_1(0) S_1 \\
 & + (1 - v_2) \lambda_2(0) \hat{A}_2^* - (1 + v_2) \lambda_2(0) \hat{S}_2^* - (1 - v_2) \lambda_2(0) \hat{A}_2 + (1 + v_2) \lambda_2(0) \hat{S}_2 \\
 & + (1 + w_1) \mu_1(0) \hat{L}_1^* - (1 - w_1) \mu_1(0) \hat{R}_1^* - (1 + w_1) \mu_1(0) \hat{L}_1 \\
 & + (1 - w_1) \mu_1(0) \hat{R}_1 + (1 + w_2) \mu_2(0) \hat{L}_2^* - (1 - w_2) \mu_2(0) \hat{R}_2^* \\
 & - (1 + w_2) \mu_2(0) \hat{L}_2 + (1 + w_2) \mu_2(0) \hat{R}_2
 \end{aligned}$$

/from conditions (30) through (37)/

$$\begin{aligned}
 (49) \geq & \eta \hat{A}_1^* - \hat{S}_1^* + \hat{A}_2^* - \hat{S}_2^* - \hat{L}_1^* + \hat{R}_1^* - \hat{L}_2^* + \hat{R}_2^* \quad / \\
 & -\eta \hat{A}_1 - \hat{S}_1 + \hat{A}_2 - \hat{S}_2 - \hat{L}_1 + \hat{R}_1 - \hat{L}_2 + \hat{R}_2 \quad /
 \end{aligned}$$

/from condition (16)/

$$(50) = 0$$

— 1 —

$$\{U(\alpha_1), \dots, U(\alpha_{n+1})\} = \{U(\alpha'_1), \dots, U(\alpha'_{n+1})\}$$

(VII) $\text{C}_6\text{H}_5\text{CH}_2\text{COCl} \rightarrow \text{C}_6\text{H}_5\text{CH}_2\text{COCl} + \text{HCl}$ (90%)

$$V_{\mu}(t,0) = \left(U^{\dagger}(t,0) \right)_{\mu}^{~~\nu} V_{\nu}(0) \left(U(0) \right)^{\dagger} = e^{-i \int_0^t d\tau H(\tau)}.$$

$$\left(\frac{d}{dx} \right)^n \left(\frac{1}{x} \right) = (-1)^n n! x^{-(n+1)} + O(x^{-n-1})$$

$$H = \frac{1}{2} \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 \right) + V_{\text{ext}}(\theta)$$

³ See, e.g., *U.S. v. Sodalis Inc.*, 750 F.2d 1032 (5th Cir. 1985), *U.S. v. Bausch & Lomb Optical Co.*, 750 F.2d 1032 (5th Cir. 1985), *U.S. v. Philip Morris Inc.*, 750 F.2d 1032 (5th Cir. 1985).

Example

In order to illustrate the basic features of an optimal solution, the upper half of Figure 1 gives a hypothetical lifetime pattern of income. This pattern of income shows a low level of income during the individual's youth, a higher level of income during middle age, and again a low level of income during old age. In this example, let us also assume that the individual's utility function is of the specific form

$$(51) \quad u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$$

Let the interest rates on debt (i_1 and i_2), the interest rates on bonds (r_1 and r_2), and the individual's rate of discount (σ) have the following relationship,

$$(52) \quad i_2 > i_1 > \theta > r_1 > r_2$$

Also let the transactions costs coefficients for bonds (w_1 and w_2) and debt (v_1 and v_2) have the following relationship,

$$(53) \quad w_2 < w_1 \text{ and } v_1 > v_2$$

For simplicity, assume that $B_1^0 = B_2^0 = D_1^0 = D_2^0 = 0$. Given these assumptions, the resulting optimal patterns of consumption, bond holdings, and indebtedness might look as depicted in Figure 1.

In this example, the individual's lifetime can be divided into ten subperiods. Some of the important characteristics of each subperiod are as follows:

$[0, t_1]$ In the first subperiod, the individual's level of consumption is greater than his income; and the deficit is financed by taking out loans of the first type ($D_1 > 0$ and $L_1 > 0$). Since these loans

where $\delta = \min\{1, \min_{i \in [n]} \mu_i^2\}$, $\alpha = \max\{1, \max_{i \in [n]} \mu_i^2\}$, $\beta = \max\{\mu_1^2, \dots, \mu_n^2\}$, $\gamma = \min\{\mu_1^2, \dots, \mu_n^2\}$.

Therefore, the empirical variance of $\hat{\mu}_i$ is bounded by $\frac{1}{n} \leq \text{Var}(\hat{\mu}_i) \leq \frac{1}{n}$ for all $i \in [n]$.

Combining the above two bounds, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n} \sum_{i=1}^n \text{Var}(\hat{\mu}_i) \leq \frac{1}{n}$.

Since $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n \hat{\mu}_i$, we have $\text{Var}(\hat{\mu}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(\hat{\mu}_i)$. This completes the proof.

Proof of Corollary 3.1. By Theorem 3.1, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.2. By Theorem 3.2, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.3. By Theorem 3.3, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.4. By Theorem 3.4, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.5. By Theorem 3.5, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.6. By Theorem 3.6, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.7. By Theorem 3.7, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.8. By Theorem 3.8, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.9. By Theorem 3.9, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.10. By Theorem 3.10, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.11. By Theorem 3.11, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.12. By Theorem 3.12, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.13. By Theorem 3.13, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.14. By Theorem 3.14, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.15. By Theorem 3.15, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.16. By Theorem 3.16, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.17. By Theorem 3.17, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

Proof of Corollary 3.18. By Theorem 3.18, we have $\text{Var}(\hat{\mu}) \leq \frac{1}{n}$. This completes the proof.

will not be paid off for a relatively long period of time (after point t_4), the high transactions costs are compensated for by the low interest charges. From conditions (24) and (26) and equation (51), the optimal pattern of consumption is defined by the differential equation

$$(54) \quad \dot{c}/c = \underline{l}(1 + w_1)i_1 - \theta \underline{l}/\sigma.$$

The level of consumption will be increasing during this subperiod since $i_1 > 0$.

$\underline{l}t_1, t_2 \bar{l}$ In the second subperiod, the individual now takes out loans of the second type because these loans will be repaid in a relatively short period of time. The higher interest charges are compensated for by the lower transactions costs. Again it is not difficult to show that consumption will be increasing during this subperiod.

$\underline{l}t_2, t_3 \bar{l}$ In the third subperiod, it is not optimal to use either type of debt to alter the consumption path; and

$$(55) \quad c = x - i_1 D_1 - i_2 D_2.$$

The gain from rearranging the pattern of consumption is not worth the combined interest charges and transactions costs.

$\underline{l}t_3, t_4 \bar{l}$ In the fourth subperiod, the level of consumption is less than income; and the surplus is used to pay back loans of the second type which have the higher interest charges ($R_2 > 0$). From conditions (24) and (28) and equation (51), the optimal path of consumption is defined by

$$(56) \quad \dot{c}/c = \underline{l}(1 - w_2)i_2 - \theta \underline{l}/\sigma.$$

Again consumption will be increasing if w_2 is not very large.

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C \left(\|f\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial f}{\partial x} \right\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial^2 f}{\partial x^2} \right\|_{L^2(\mathbb{R}^d)} \right) \quad (1)$$

$$|\nabla u|^2 \leq C_1 \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \right) \quad \text{and} \quad |\nabla^2 u|^2 \leq C_2 \left(|u|^2 + |\nabla u|^2 \right) \quad (2)$$

$$|\nabla^2 u|^2 \leq C_3 \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \right) \quad \text{and} \quad |\nabla^3 u|^2 \leq C_4 \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \right) \quad (3)$$

$$|\nabla^k u|^2 \leq C_k \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 + \dots + |\nabla^{k-1} u|^2 \right)$$

$$(4)$$

$$\text{Hence, applying H\"older's inequality, we get} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (5)$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (6)$$

$$\text{Since } u \in C_c^\infty(\mathbb{R}^d), \text{ we know that } \nabla^k u \in C_c^\infty(\mathbb{R}^d) \text{ for all } k \geq 0. \text{ Hence, } \left\| \nabla^k u \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ Therefore, } (6) \text{ holds.}$$

$$\text{Thus, we have shown that } \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ This completes the proof of Lemma 2.1.} \quad \square$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (7)$$

$$(8)$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (9)$$

$$\text{Since } u \in C_c^\infty(\mathbb{R}^d), \text{ we know that } \nabla^k u \in C_c^\infty(\mathbb{R}^d) \text{ for all } k \geq 0. \text{ Hence, } \left\| \nabla^k u \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ Therefore, } (9) \text{ holds.}$$

$$\text{Thus, we have shown that } \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ This completes the proof of Lemma 2.1.} \quad \square$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (10)$$

$$(11)$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (12)$$

$$\text{Since } u \in C_c^\infty(\mathbb{R}^d), \text{ we know that } \nabla^k u \in C_c^\infty(\mathbb{R}^d) \text{ for all } k \geq 0. \text{ Hence, } \left\| \nabla^k u \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ Therefore, } (12) \text{ holds.}$$

$$\text{Thus, we have shown that } \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ This completes the proof of Lemma 2.1.} \quad \square$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (13)$$

$$(14)$$

$$\text{Therefore, we have} \quad \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} \leq C_k \left(\left\| u \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla u \right\|_{L^2(\mathbb{R}^d)} + \dots + \left\| \nabla^{k-1} u \right\|_{L^2(\mathbb{R}^d)} \right) \quad (15)$$

$$\text{Since } u \in C_c^\infty(\mathbb{R}^d), \text{ we know that } \nabla^k u \in C_c^\infty(\mathbb{R}^d) \text{ for all } k \geq 0. \text{ Hence, } \left\| \nabla^k u \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ Therefore, } (15) \text{ holds.}$$

$$\text{Thus, we have shown that } \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2(\mathbb{R}^d)} < \infty \text{ for all } k \geq 0. \text{ This completes the proof of Lemma 2.1.} \quad \square$$

/t₄, t₅/ In the fifth subperiod, the surplus is used to pay back loans of the first type; and at $t = t_5$, all loans have been repaid.

/t₅, t₆/ In the sixth subperiod, consumption is still less than income; and the individual begins to save for his old age. The surplus is invested in bonds of the first type. Because these bonds will be held for a relatively long period of time, the high transactions costs are compensated for by the higher interest income. From conditions (19) and (20) and equation (51), the optimal path of consumption is defined by

$$(57) \quad \dot{c}/c = \bar{I}(1 - v_1)r_1 - \theta\bar{T}/\sigma.$$

Consumption will be decreasing during this subperiod.

/t₆, t₇/ In the seventh subperiod, the surplus is now invested in bonds of the second type because of the lower transactions costs.

/t₇, t₈/ In the eighth subperiod, it is not optimal to save at all because of the high transactions costs relative to the small amount of interest that could be earned on either type of bond during this short period and

$$(58) \quad c = x + r_1 B_1 + r_2 B_2.$$

/t₈, t₉/ In the ninth subperiod, the individual sells his bonds of the second type in order to pay for consumption during his old age. From conditions (19) and (22) and equation (51), consumption is defined by

$$(59) \quad \dot{c}/c = \bar{I}(1 + v_2)r_2 - \theta\bar{T}/\sigma.$$

/t₉, T/ In the last subperiod, the individual sells his stock of bonds of the first type to pay for consumption. At point $t = T$, the stock of bonds is exhausted.

$$\psi \approx \frac{1}{2} \ln \left(\frac{\rho}{\rho_0} \right) + \frac{1}{2} \left(\frac{\rho}{\rho_0} - 1 \right)^2 \frac{\partial^2 \psi}{\partial \rho^2} \Big|_{\rho=\rho_0} \quad (1)$$

where ρ_0 is the value of ρ at the center of the star.

It is also assumed that the density profile is spherically symmetric and depends only on the radial coordinate r .

The total energy of the star is given by the expression

$$E = \frac{1}{2} \int dV \left[\frac{1}{2} \rho v^2 + \frac{1}{2} \rho c_s^2 \nabla^2 \psi + \frac{1}{2} \rho g^2 + \frac{1}{2} \rho \omega^2 r^2 \right] \quad (2)$$

where v is the velocity, c_s is the sound speed, g is the gravitational constant, and ω is the angular velocity.

It is also assumed that the star is in hydrostatic equilibrium, which means that the pressure gradient is balanced by the gravitational force.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

where G is the gravitational constant and M is the mass of the star.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

where G is the gravitational constant and M is the mass of the star.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

where G is the gravitational constant and M is the mass of the star.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

where G is the gravitational constant and M is the mass of the star.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

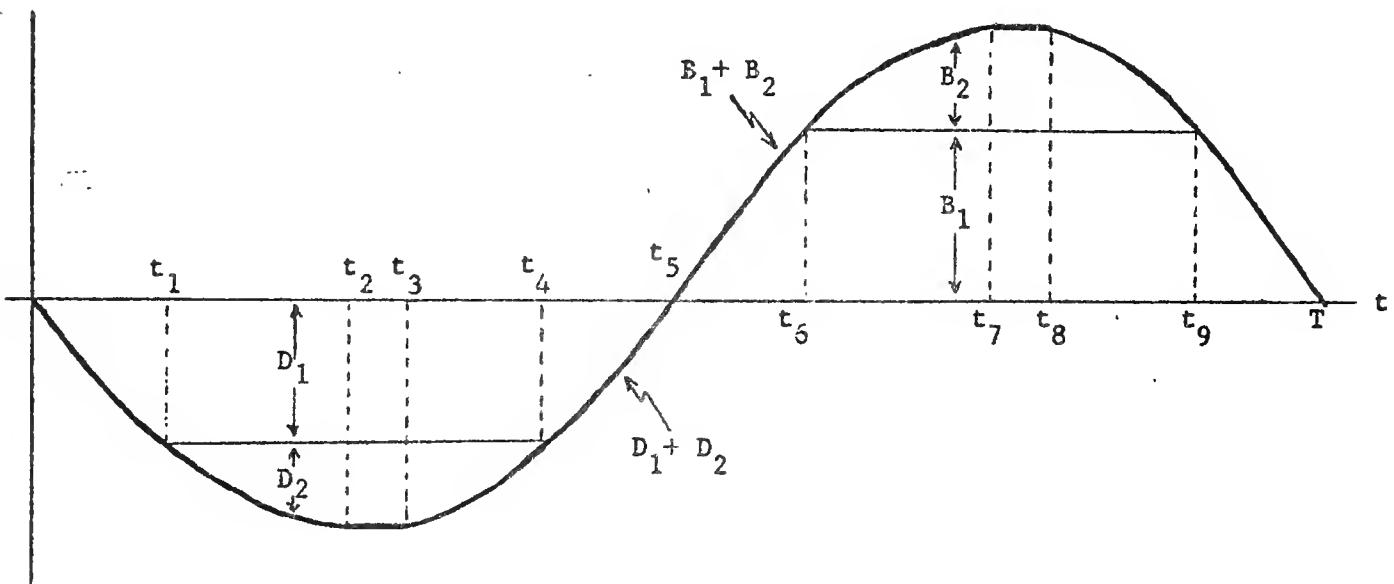
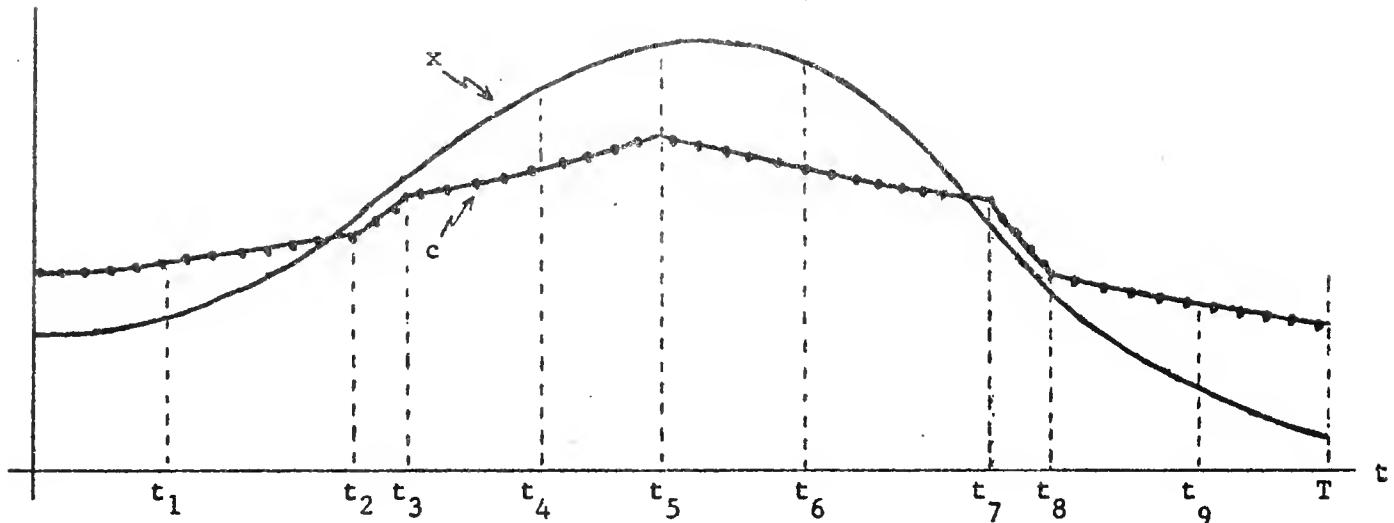
where G is the gravitational constant and M is the mass of the star.

The equations of motion for the star are given by the expressions

$$\frac{d}{dr} \left(\rho v \right) = - \frac{G M}{r^2} \rho \quad (3)$$

FIGURE 1

Optimal Patterns of Consumption, Bonds Holdings, and Indebtedness
for a Typical Life-Time Pattern of Income





UNIVERSITY OF ILLINOIS-URBANA



3 0112 060296883